FRG for BEC-BCS crossover

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Renormalisation Flow and Universality for Ultracold Fermionic Atoms
Phys. Rev. A77, 053627 (2007), cond-mat/0701198

Functional Renormalisation Group Approach to the BCS-BEC Crossover
Annalen der Physik 522, 615-656 (2010), arXiv:09017.2193

$$S[\varphi,\psi] = \int_0^\beta d\tau \int d^3x \Big(\psi^\dagger \left(\partial_\tau - \Delta - \mu\right)\psi + \varphi^* \left(\partial_\tau - \frac{1}{2}\Delta + \nu - 2\mu\right)\varphi - \frac{h_\varphi}{2}\left(\varphi^*\psi^T\epsilon\psi - \varphi\psi^\dagger\epsilon\psi^*\right)\Big),$$

ψ two-component non-rel. spinor field (two hyperfine levels of a fermionic atom)

φ complex scalar field for a dimer of two atoms, strongly coupled molecule in BEC or weakly correlated Cooper pair in BCS phase

μ chemical potential

ν detuning from Feshbach-resonance ~ $B-B_0$

$B < B_0$ BEC condensate of tightly bound molecules, finite part in particle density, positive scattering length in atom-atom scattering denoted *a*

$B > B_0$ BCS condensate of loosely correlated Cooper pairs, negative *a*

$B = B_0$ Feshbach resonance, $a^{-1}=0$

FRG: from normal to broken U(1) symmetry phase in the range $-\infty < \mu < \infty$

On the SSB side: smooth crossover from BEC ($\mu < 0$) to BCS ($\mu > 0$) regime

The effective potential and its RGE

$$p(T,\mu) = -U(\rho_0; T, \mu).$$

$$\begin{aligned} n &= \left.\frac{\partial p}{\partial \mu}\right|_T = -\frac{\partial U}{\partial \mu}(\rho_0), \\ s &= \left.\frac{\partial p}{\partial T}\right|_\mu = -\frac{\partial U}{\partial T}(\rho_0), \\ \epsilon &= -p + Ts + \mu n = U(\rho_0) + Ts + \mu n. \end{aligned}$$

Thermodynamics determined by the infrared value of the effective potential $U_{k=0}$ in the condensate ρ_0

Initial expression for the potential in the UV:

$$U_{k\to\infty}(\rho) = U_{\rm cl}(\rho) = (\nu - 2\mu)\rho, \quad \rho = \varphi^*\varphi.$$

$$\begin{aligned} \partial_k U_k(\varphi) = \frac{1}{2}\int_q \Big\{ &{\rm tr}_\varphi [G_\varphi^{-1}(\varphi;q) + R_k^{(\varphi)}(q)]^{-1}\partial_k R_k^{(\varphi)}(q) \\ &-{\rm tr}_\psi [G_\psi^{-1}(\varphi;q) + R_k^{(\psi)}(q)]^{-1}\partial_k R_k^{(\psi)}(q) \Big\}. \end{aligned}$$

Bosonic

Fermionic

IR-regulated contributions to the RG-rate of U

Ansatz for the effective action:

$$\Gamma_k = \int_{\tau,\vec{x}} \Big\{ \bar{U}_k(\bar{\rho}) + \bar{\varphi}^*(Z_\varphi \partial_\tau - \frac{1}{2} A_\varphi \Delta)\bar{\varphi} + \psi^\dagger(\partial_\tau - \Delta - \mu)\psi - \frac{\bar{h}_\varphi}{2}(\bar{\varphi}^* \psi^T \epsilon \psi - \bar{\varphi}\psi^\dagger \epsilon \psi^*)\Big\}. \quad (13)$$

$$\Gamma_k = \int_{\tau,\vec{x}} \Big\{ U_k(\rho) + \varphi^*(S_\varphi \partial_\tau - \frac{1}{2}\Delta)\varphi + \psi^\dagger(\partial_\tau - \Delta - \mu)\psi - \frac{h_\varphi}{2}(\varphi^* \psi^T \epsilon \psi - \varphi\psi^\dagger \epsilon \psi^*)\Big\}. \quad (14)$$

After field rescaling one finds (14)

$$\varphi = A^{1/2}\bar{\varphi}, \qquad S_\varphi = Z_\varphi / A_\varphi, \qquad h_\varphi = \bar{h}_\varphi / \sqrt{A_\varphi}$$

$$k\partial_k U_k = \eta_{A_\varphi}\,\rho\, U_k' + \frac{\sqrt{2}k^5}{3\pi^2 S_\varphi}\left(1 - 2\eta_{A_\varphi}/5\right) s_{\mathrm{B}}^{(0)} - \frac{k^5}{3\pi^2}\, l(\tilde{\mu})\, s_{\mathrm{F}}^{(0)},$$

$W_{1,2}$ scaled boson masses

W_3 scaled fermion mass

$$s_B^{(0)} = \left[\sqrt{\frac{1+w_1}{1+w_2}} + \sqrt{\frac{1+w_2}{1+w_1}}\right] \times \left[\frac{1}{2} + N_{\mathrm{B}}(\sqrt{1+w_1}\sqrt{1+w_2}/S_\varphi)\right],$$

$$s_{\mathrm{F}}^{(0)} = \frac{2}{\sqrt{1+w_3}}\left[\frac{1}{2} - N_{\mathrm{F}}(\sqrt{1+w_3})\right].$$

Ansatz for U_k:

$$U_k = -p_k + m_\varphi^2(\rho - \rho_0) + \frac{1}{2}\lambda_\varphi(\rho - \rho_0)^2 - n_k\delta\mu + \alpha_k(\rho - \rho_0)\delta\mu.$$

Fermion density defined through derivation with respect to δμ and setting δμ=0.

Superfluidity transition

O(2) universality class

$$\text{Symmetric phase}: \quad \rho_0 = 0, \quad m_\varphi^2 > 0,$$
$$\text{Symmetry broken phase}: \quad \rho_0 > 0, \quad m_\varphi^2 = 0,$$
$$\text{Phase transition}: \quad \rho_0 = 0, \quad m_\varphi^2 = 0,$$

Fixing the value of *a* for $T < T_c$ running *k*, ρ_0 appears at some k_{SSB} and stays till $k=0$

for $T > T_c$ ρ_0 is present for $k_{SSB} > k > k_{SR}$ (local order)

BEC side of the crossover large value of k_{SSB} substantial binding energy, high T_c

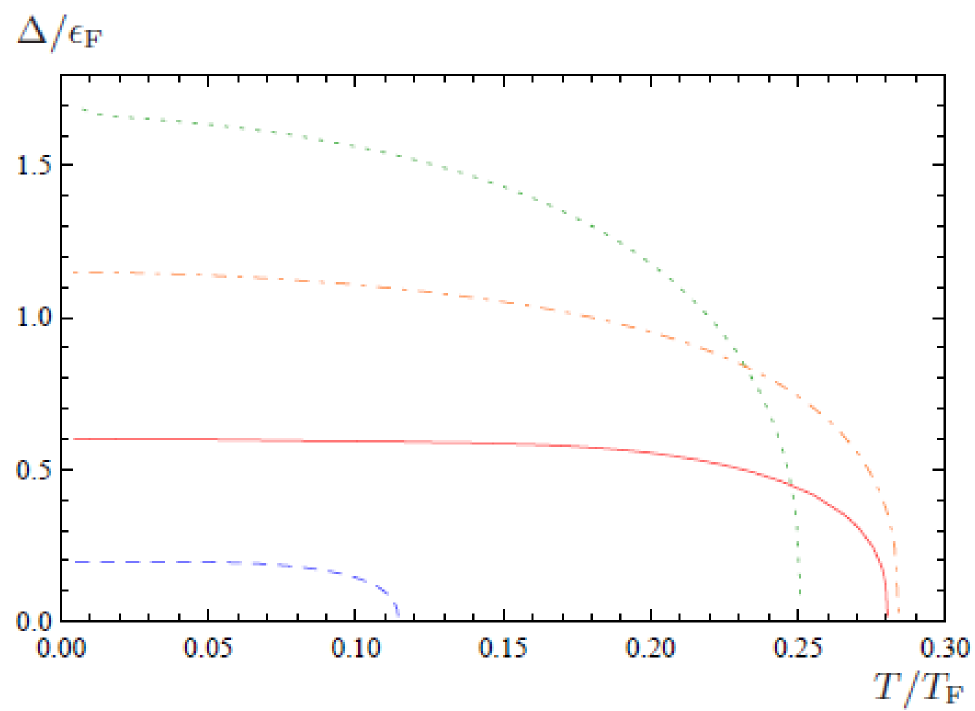
BCS very low k_{SSB} loosely correlated Cooper pairs, low T_c

Only fermionic fluctuations: first order, bosonic contributions smoothing it into a continuous transition

$\Delta = h_\varphi \rho_0$ superconducting gap

k_F $n = k_F^3/(3\pi^2)$, $\epsilon_F = k_F^2/2M = T_F$

$$\partial_k n_k = -\partial_k \frac{\partial}{\partial \mu} U_k$$



Relating relevant couplings ($\bar{m}_\varphi^2$, h_φ) to observable quantities of the two-body interaction

The vacuum problem: two-fermion scattering problem, *n=T=0,* $\rho_0=0$ *(q.m.)*

Two-point functions:

$$
\begin{aligned}
P_{\rm F}(q) &= iq_0 + \vec{q}^{\,2} - \mu, \\
P_\varphi(q) &= iS_\varphi q_0 + \vec{q}^{\,2}/2 + m_\varphi^2.
\end{aligned}
$$

$M_\psi = -\mu$
$M^2_\varphi = m^2_\varphi(\mu, B)$

RGE:

$$
\begin{aligned}
\partial_k \bar{m}_\varphi^2 &= \frac{\bar{h}_\varphi^2}{6\pi^2 k^3}\theta(k^2+\mu)(k^2+\mu)^{3/2}, \\
\partial_k Z_\varphi &= -\frac{\bar{h}_\varphi^2}{6\pi^2 k^5}\theta(k^2+\mu)(k^2+\mu)^{3/2}, \\
\partial_k A_\varphi &= -\eta_{A_\varphi} A_\varphi / k = -\frac{\bar{h}_\varphi^2}{6\pi^2 k^5}\theta(k^2+\mu)(k^2+\mu)^{3/2},
\end{aligned}
$$

h_φ is not running, analytic integration of first order equations, down from k=Λ

Initial condition: $$\bar{m}_\varphi^2(\Lambda) = \nu(B) - 2\mu + \delta\nu(\Lambda).$$

BCS-side at *k=0, μ=0* $$\bar{m}_{\varphi,0}^2 = \bar{m}_{\varphi,\Lambda}^2 - \frac{\bar{h}_{\varphi,\Lambda}^2}{6\pi^2}\Lambda = \mu_{\rm M}(B-B_0) + \delta\nu(\Lambda) - \frac{\bar{h}_{\varphi,\Lambda}^2}{6\pi^2}\Lambda$$

At the resonance (B = B_0) the boson mass vanishes: $$\delta\nu(\Lambda) = \frac{\bar{h}_{\varphi,\Lambda}^2}{6\pi^2}\Lambda.$$

Definition of the scattering length from excchange of the scalar between two fermions

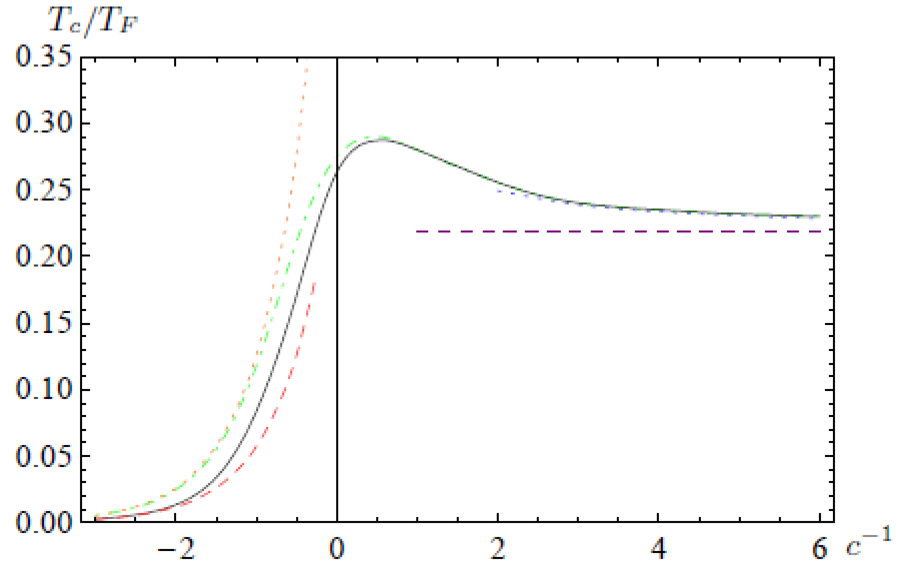
$$a = \frac{\lambda_{\psi,\mathrm{eff}}}{8\pi}. \qquad \lambda_{\psi,\mathrm{eff}} = -\frac{\bar{h}^2_{\varphi,\Lambda}}{\bar{P}_\varphi(\omega, \vec{p}^2 = 0, \mu)}.$$

$$a = -\frac{\bar{h}^2_{\varphi,\Lambda}}{8\pi\mu_\mathrm{M}(B - B_0)}.$$

B-B₀, *a:* observables

Same form on the BEC-side, **initial choice of $m_\varphi^2(\Lambda)$, h_φ, translated into a^{-1} and B**

$$c \equiv a k_F$$



Binding energy of the molecule: $M_\varphi - 2M_\psi = 2\mu$

Free fermion mass: $-\mu$, Meson (mass)^2: $m_\varphi^2/Z_\varphi = 0$

Fixed point structure

$$\partial_t m_\varphi^2 = \frac{k}{6\pi^2} h_\varphi^2 + \eta_{A_\varphi} m_\varphi^2,$$
$$\partial_t S_\varphi = -\frac{1}{6\pi^2 k} h_\varphi^2 + \eta_{A_\varphi} S_\varphi,$$
$$\partial_t h_\varphi^2 = \eta_{A_\varphi} h_\varphi^2,$$
$$\eta_{A_\varphi} = -\frac{\partial_t A_\varphi}{A_\varphi} = \frac{1}{6\pi^2 k} h_\varphi^2.$$

Dimensionless combinations:

$$\partial_t \left(\frac{h_\varphi^2}{k}\right) = (-1 + \eta_{A_\varphi}) \frac{h_\varphi^2}{k} \qquad \partial_t \left(\frac{m_\varphi^2}{k^2}\right) = \frac{1}{6\pi^2} \frac{h_\varphi^2}{k} - (2 - \eta_{A_\varphi}) \frac{m_\varphi^2}{k^2} = 1 - \frac{m_\varphi^2}{k^2},$$

$$\partial_t S_\varphi = (S_\varphi - 1)\eta_{A_\varphi}.$$

$\frac{h_\varphi^2}{k}$ Its large value corresponds to „broad Feshbach resonance”

The above fixed point controls the crossover of this case also for $n \neq 0,$ $T \neq 0$